Warsaw University of Technology
Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

Variable conditioning of the system - an example of two springs

$$[K]\cdot\{q\}=\{F\}$$

$$[K+\delta K]\{q+\delta q\}=\{F+\delta F\}$$

relative error of the global vector of nodal parameters:

$$\frac{\|\{\delta q\}\|}{\|\{q\}\|}\leq\underbrace{\|[K]\|\cdot\|[K]^{-1}\|}_{\text{Cond}[K]}\cdot\left(\frac{\|\{\delta F\}\|}{\|\{F\}\|}+\frac{\|[\delta K]\|}{\|[K]\|}\right)$$

J. Steer.

Condition number:

$$\boxed{\text{Cond}[K]=\frac{\text{change of solution}}{\text{change of input data}}}$$

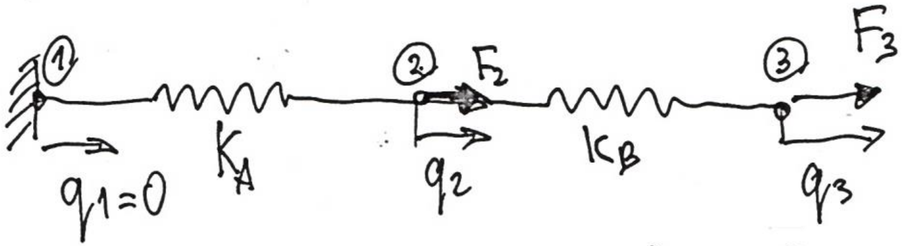
$\text{cond}[K] \approx 1$ – problem well-conditioned

$\text{cond}[K] \gg 1$ – problem ill-conditioned

(great differences between FEs stiffnesses, unstable boundary conditions)

	vector	matrix
Euclidean norm L_2	$\\|\{q\}\\|_2 = \sqrt{\sum_i (q_i)^2}$	$\\|[K]\\|_2 = \sqrt{\sum_j \sum_i (k_{ij})^2}$
Maximum norm L_∞	$\\|\{q\}\\|_\infty = \max_i \|q_i\|$	$\\|[K]\\|_\infty = \max_i \left(\sum_j \|k_{ij}\|\right)$

EXAMPLE:



NDOF = 3

NOF = 1

N = 3 - 1 = 2

$$\underset{1\times3}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3 \rfloor \quad , \quad \underset{2\times2}{[k]_e} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad , \quad \underset{1\times3}{\lfloor F \rfloor} = \lfloor R_1, F_2, F_3 \rfloor$$

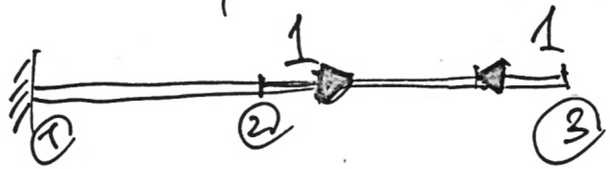
$$\underset{3\times3}{[K]} = \begin{bmatrix} k_A & -k_A & 0 \\ -k_A & k_A+k_B & -k_B \\ 0 & -k_B & k_B \end{bmatrix} + \begin{array}{l} \text{Boundary} \\ \text{Conditions} \\ (q_1 = 0) \end{array}$$

$$\begin{bmatrix} k_A+k_B & -k_B \\ -k_B & k_B \end{bmatrix} \cdot \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \quad , \quad \underset{2\times1}{\{q\}} = \underset{2\times2}{[K]^{-1}} \cdot \underset{2\times1}{\{F\}}$$

$$[K]^{-1} = \frac{1}{\det[K]} \cdot [K^P]^T = \frac{\begin{bmatrix} k_B & k_B \\ k_B & k_A+k_B \end{bmatrix}^T}{(k_A+k_B)k_B - (-k_B)\cdot(-k_B)} = \frac{1}{k_A \cdot k_B} \begin{bmatrix} k_B & k_B \\ k_B & k_A+k_B \end{bmatrix}$$

Lets assume:

$$F_2 = 1\,N \quad , \quad F_3 = -1\,N$$ (forces being in equilibrium)



$$\delta F_2 = -0.001\,N \quad , \quad \delta F_3 = 0$$

$$F_2 + \delta F_2 = 0.999\,N \quad , \quad F_3 + \delta F_3 = -1\,N$$

$$\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{K_A} & \frac{1}{K_A} \\ \frac{1}{K_A} & \frac{1}{K_A} + \frac{1}{K_B} \end{bmatrix} \cdot \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$q_2 = \frac{F_2}{K_A} + \frac{F_3}{K_A} \quad (1), \qquad q_3 = \frac{F_2}{K_A} + F_3 \left(\frac{1}{K_A} + \frac{1}{K_B} \right) \quad (2),$$

$$q_2 + \delta q_2 = \frac{F_2 + \delta F_2}{K_A} + \frac{F_3 + \delta F_3}{K_A} \quad (3), \qquad q_3 + \delta q_3 = \frac{F_2 + \delta F_2}{K_A} + (F_3 + \delta F_3) \cdot \left(\frac{1}{K_A} + \frac{1}{K_B} \right) \quad (4),$$

$$\delta q_2 = (q_2 + \delta q_2) - q_2 \quad (5), \qquad \delta q_3 = (q_3 + \delta q_3) - q_3 \quad (6)$$

Euclidean norms:

$$\|\{q\}\|_2 = \sqrt{q_2^2 + q_3^2} \; ; \qquad \|\{\delta q\}\|_2 = \sqrt{\delta q_2^2 + \delta q_3^2}$$

$$\|\{F\}\|_2 = \sqrt{F_2^2 + F_3^2} \, , \qquad \|\{\delta F\}\|_2 = \sqrt{\delta F_2^2 + \delta F_3^2}$$

$$\frac{\|\{\delta F\}\|_2}{\|\{F\}\|_2} = 0.7071 \cdot 10^{-3}$$

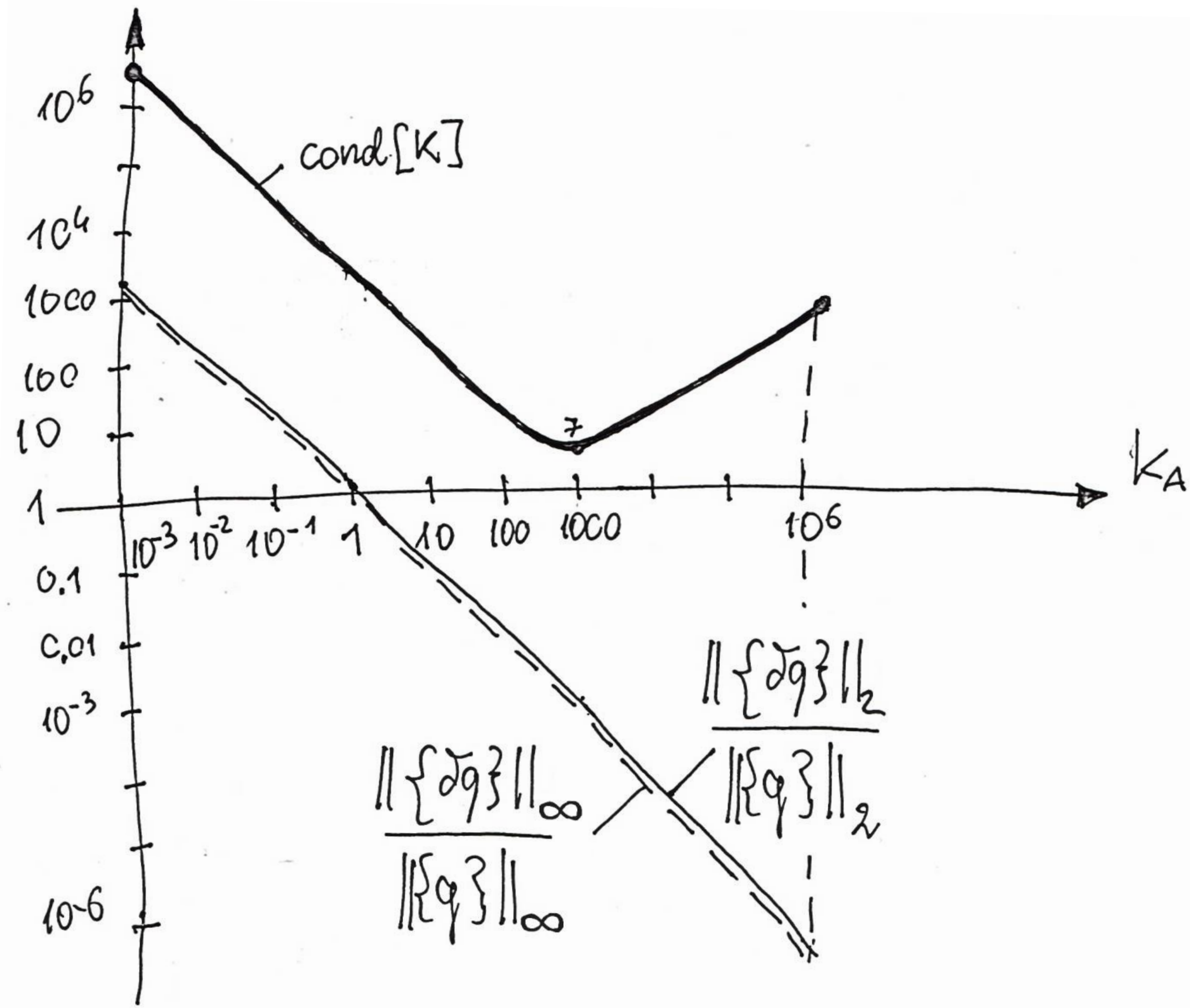
Lets assume: $K_B = \text{const} = 1000 \, \frac{N}{mm}$

[N/mm] K_A	(1) q_2	(2) q_3	(3) $q_2+\delta q_2$	(4) $q_3+\delta q_3$	(5) δq_2	(6) δq_3	$\frac{\|\{\delta q\}\|_2}{\|\{q\}\|_2}$	$\text{cond}[K]$	$\text{cond}[K]\cdot\frac{\|\{\delta F\}\|_2}{\|\{F\}\|_2}$
0.001	0	-0.001	-1	-1.001	-1	-1	1414.21	$4\cdot10^6$	2828.43

$$\frac{\| \{\delta F\} \|_2}{\| \{F\} \|_2} = 0.7071 \cdot 10^{-3}$$

Lets assume: $K_B = \text{const} = 1000 \frac{N}{mm}$

[N/mm] K_A	(1) q_2	(2) q_3	(3) $q_2+\delta q_2$	(4) $q_3+\delta q_3$	(5) δq_2	(6) δq_3	$\frac{\|\{\delta q\}\|_2}{\|\{q\}\|_2}$	cond [K]	$\text{cond}[K] \cdot \frac{\|\{\delta F\}\|_2}{\|\{F\}\|_2}$
0.001	0	-0.001	-1	-1.001	-1	-1	1414.21	$4 \cdot 10^6$	2828.43
1	0	-0.001	-0.001	-0.002	-0.001	-0.001	1.41	$4 \cdot 10^3$	2.83
1000	0	-0.001	-10^{-6}	$-1.001 \cdot 10^{-3}$	-10^{-6}	-10^{-6}	$1.41 \cdot 10^{-3}$	7	0.00495
10^6	0	-0.001	-10^{-9}	$-1.000 \cdot 10^{-3}$	-10^{-9}	-10^{-9}	$1.41 \cdot 10^{-6}$	$\underline{\underline{1000}}$	0.7

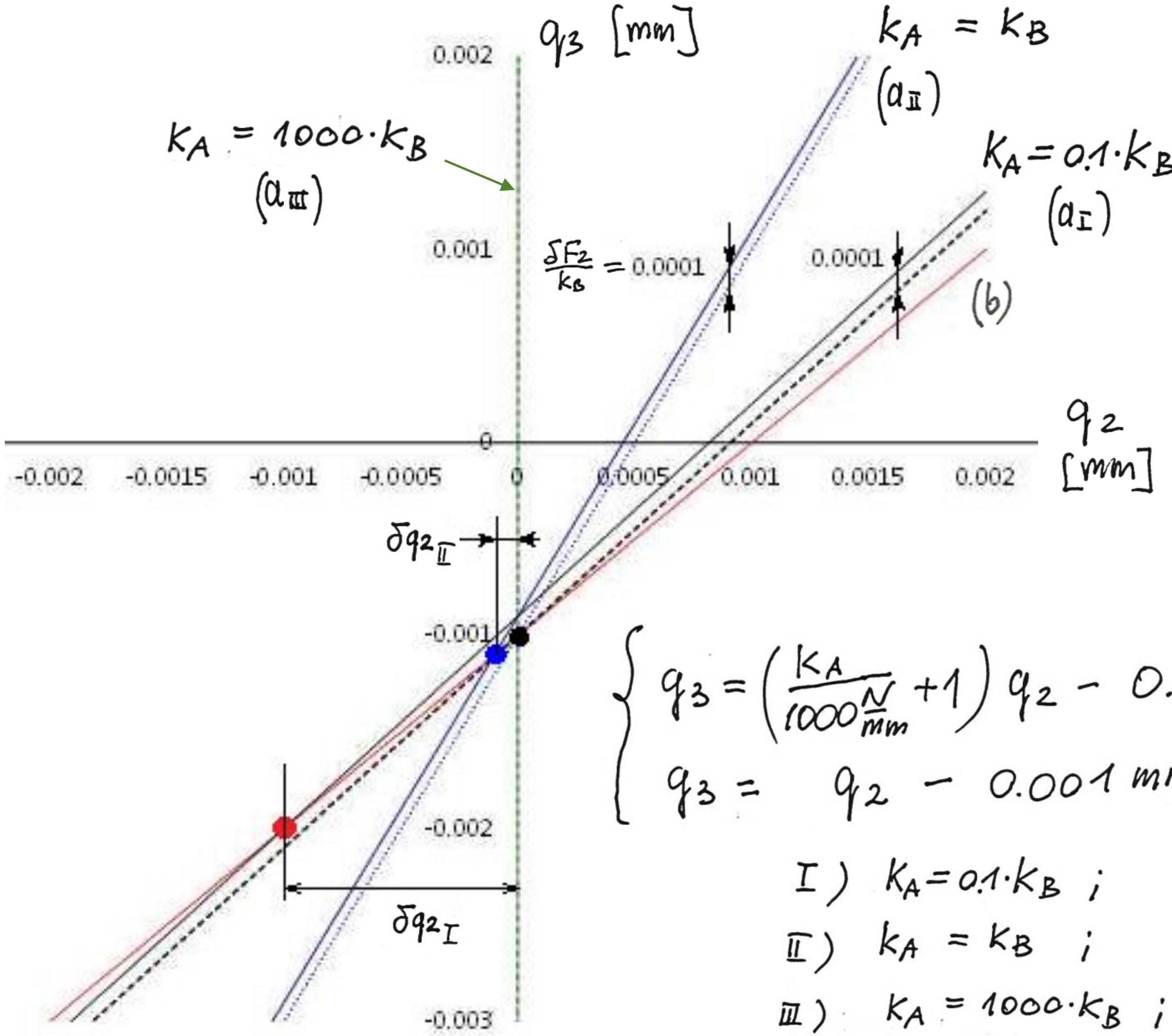

10^6
10^4
1000
100
10
1
0.1
0.01
10^{-3}
10^{-6}
10^{-3} 10^{-2} 10^{-1} 1 10 100 1000 10^6
K_A
cond[K]
7
$\frac{\|\{\partial q\}\|_2}{\|\{q\}\|_2}$
$\frac{\|\{\partial q\}\|_\infty}{\|\{q\}\|_\infty}$

$$\begin{bmatrix} k_A + k_B & -k_B \\ -k_B & k_B \end{bmatrix} \cdot \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{cases} (k_A + k_B)\, q_2 - k_B\, q_3 = F_2 & \Rightarrow q_3 \\ -k_B\, q_2 + k_B \cdot q_3 = F_3 & \Rightarrow q_3 \end{cases}$$

$$\begin{cases} q_3 = \dfrac{k_A + k_B}{k_B} q_2 - \dfrac{F_2}{k_B} \\ q_3 = q_2 + \dfrac{F_3}{k_B} \end{cases}$$

FOR : $F_2 = 1N$, $F_3 = -1N$, $k_B = 1000 \frac{N}{mm}$



$$
\begin{cases}
q_3 = \left(\dfrac{K_A}{1000 \frac{N}{mm}} + 1 \right) q_2 - 0.001\,mm & (a) \\
q_3 = q_2 - 0.001\,mm & (b)
\end{cases}
$$

I) $K_A = 0.1 \cdot K_B$;

II) $k_A = k_B$;

III) $K_A = 1000 \cdot K_B$;